Relativistic spin-1 bosons in a magnetic field

1993 J. Phys. A: Math. Gen. 26 1397

(http://iopscience.iop.org/0305-4470/26/6/021)

J. Phys. A: Math. Gen. 26 (1993) 1397–1408. Printed in the UK

Relativistic spin-1 bosons in a magnetic field

J Daicic and N E Frankel
School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia

Received 30 June 1992

Abstract. The quantum mechanics of charged, massive, spin-1 bosons in the presence of a homogeneous magnetic field (HMF) is studied using a six-component wavefunction formalism. The energy eigenvalues are compared with those previously obtained via other formalisms, the equations of motion of certain operators are given, and the positive and negative energy eigensolutions are obtained by the use of a ladder operator method. The six-component current for the case of general external electromagnetic fields is also displayed and finally, the employment of the eigensolutions and current in a study of a spin-1 boson–antiboson plasma is discussed.

1. Introduction

In a previous paper [1], the present authors developed a study of the relativistic spin-1 boson plasma with no external fields present. This work included an investigation of the solution of the Proca equations [2] for free spin-1 bosons, in the six-component formalism of Sakata and Taketani [3]. The aim of the present paper is to extend the study to the quantum mechanics of relativistic spin-1 bosons in external fields, in particular, a homogeneous magnetic field (HMF).

We begin with a review of previous work on this topic. Corben and Schwinger [4], followed by Tsai and co-workers [5–7], Krase *et al* [8], Weaver [9], and Vijayalakshmi *et al* [10] have attempted to find the energy eigenvalues of a relativistic spin-1 particle in external fields, in some cases with the inclusion of non-minimal electromagnetic couplings in the equations of motion. These authors employ a variety of spin-1 formalisms (see our review in section 1 of [1]), and in the papers of Tsai and co-workers [5–7] and Vijayalakshmi *et al* [10], a comparison of the energy eigenvalues obtained from the differing formalisms is made.

As we shall discuss in section 3 of the paper, the inclusion of an anomalous magnetic moment coupling, so that the gyromagnetic ratio of the spin-1 boson has a value of $g = 2$, simplifies the eigenvalues to the familiar form of solution of a relativistic particle in a HMF, as given by Johnson and Lippmann [11] in the case of the Dirac particle. Weaver [9] and Durand [12] refer to this magnetic moment as the 'natural' one for bosons in the presence of external fields. In this paper, we present a new treatment of the quantum mechanics of this system by following a canonical equations-of-motion procedure for obtaining the diagonalized Hamiltonian, energy eigenvalues and wavefunctions.

Following the method developed by Johnson and Lippmann [11] for relativistic spin-$\frac{1}{2}$ fermions, we present in section 3, for the first time, the explicit forms for the six-component wavefunctions in the presence of a HMF, in the case of both positive

0305-4470/93/061397+12$07.50

energy (particle) and negative energy (antiparticle) solutions of the Sakata–Taketani equation. A new expression for the single-particle six-component current in the case of general external electromagnetic fields is also obtained.

A treatment of the solutions of the Klein–Gordon equation for spinless particles in a HMF using a two-component formalism is given by Witte *et al* [13], and these solutions are later employed in a detailed study of the spin-0 boson–antiboson plasma by Witte *et al* [14]. Similarly, the spin-$\frac{1}{2}$ solutions of Johnson and Lippmann [11] are employed by Kowalenko *et al* [15] in the response theory of a spin-$\frac{1}{2}$ fermion–antifermion plasma in a HMF. We discuss in section 4 how the wavefunctions and current of section 3 could be employed in a self-consistent random-phase approximation (RPA) treatment of a spin-1 boson–antiboson plasma in a HMF. This would be the extension of our study in [1] of a relativistic spin-1 boson plasma in no external fields.

2. Review of spin-1 bosons in external fields

Corben and Schwinger [4] were the first to present a general theory for massive particles of spin 1 and arbitrary magnetic moment. They did so by generalizing the equations of Proca [2], and were able to study the motion of such particles in an external Coulomb field. They also found the current and stress–energy tensor for this system, employing the vector formalism for the wavefunctions. Their work is fundamental, and provides the basis for many other studies. Of particular utility is the work of Young and Bludman [16], who further generalize the Proca equations to the case of explicit arbitrary anomalous quadrupole couplings, and proceed to find the Sakata–Taketani six-component Hamiltonian for this generalized case. It is this Hamiltonian, for the specific case of no anomalous quadrupole coupling, a magnetic g-factor of 2, and the external field being a HMF, which we shall employ in section 3.

Young and Bludman [16] derive this Hamiltonian by a different technique to that of Heitler [17], who derives a Hamiltonian for the case of minimal coupling only. Young and Bludman reduce the Proca equations into a six-component spinor equation by explicitly eliminating the dynamically redundant components of the spin-1 fields. Heitler, however, employs a technique whereby the Proca equations are written in the first-order 10-component Duffin–Kemmer [18] form:

$$(\beta^{\mu}\pi_{\mu} - im)\,\psi = 0 \tag{2.1}$$

where π_{μ} is the canonical 4-momentum, and the β_{μ} are 10×10 matrices, with the property

$$\beta_{\mu}\beta_{\nu}\beta_{\lambda} + \beta_{\lambda}\beta_{\nu}\beta_{\mu} = \delta_{\mu\nu}\beta_{\lambda} + \delta_{\lambda\nu}\beta_{\mu}\,. \tag{2.2}$$

A technique known as Pierce decomposition is employed to remove the dynamically redundant components of the wavefunction ψ, by separating the Duffin–Kemmer ring (defined by (2.2) above), into two sub-rings, one of which contains only the dynamically redundant field components, and the other yielding the minimally coupled Sakata–Taketani equation, and associated operators. However, this technique is far more difficult to employ when the anomalous moments are incorporated into the equations of motion.

None the less, it is possible to obtain the appropriate generalization of the Duffin–Kemmer equation (2.1) with an anomalous magnetic moment coupling γ. This is given by Vijayalakshmi *et al* [10] and is

$$(\beta^\mu \pi_\mu - \mathrm{i}m + R(x))\, \psi = 0 \tag{2.3}$$

where

$$R(x) = -\frac{e\gamma}{m} \left(\sigma_{\mu\nu} - \frac{1}{4} \{ \beta_\alpha \beta_\alpha, \sigma_{\mu\nu} \} \right) F^{\mu\nu} \tag{2.4}$$

and $\sigma_{\mu\nu} = [\beta_\mu, \beta_\nu]$.

It is of interest, given the superficial similarity of (2.1) to the Dirac equation, that the anomalous magnetic moment coupling is not of the simple $\sigma_{\mu\nu}F^{\mu\nu}$ form of the spin-$\frac{1}{2}$ case.

Durand [12] has proposed that the elimination of this latter term leads to new equations of motion which have a different anomalous moment coupling to the generalized Proca equation of Corben and Schwinger [4]. However, Vijayalakshmi *et al* [10] show that with the choice of $R(x)$ given in (2.3) that causality is not violated, contrary to the case where only the $\sigma_{\mu\nu}F^{\mu\nu}$ term is kept.

Such a choice of couplings has its own pathologies, the most serious problem being the appearance of complex energy eigenvalues for magnetic fields of higher than a critical magnitude, as demonstrated by Tsai and co-workers [5–7] and Velo and Zwangzinger [19]. Furthermore, the anomalous magnetic moment induces electric quadrupole terms in the Hamiltonian and current, when the external fields are fully generalized to include time-dependent and (possibly) inhomogeneous electric and magnetic fields.

The energy eigenvalues of the non-minimally coupled spin-1 particle in a HMF are studied by Krase *et al* [8] using the six-component Shay–Good formalism [20] for spin-1 particles, which is distinct from the Sakata–Taketani formalism (see section 2 of [1]). These authors obtain for one branch of the energy eigenvalues (for $g = 2$, i.e. $\gamma = 1$)

$$E^2_{n,s,p_z} = p_z^2 c^2 + m^2 c^4 + e\hbar c B\,(2n - 2s + 1)\,. \tag{2.5}$$

where n is a Landau-level quantum number.

This is similar to the result obtained by Johnson and Lippmann [11] for the Dirac particle in a B-field, save for the factor of 2 premultiplying the spin-projection quantum number s. Krase *et al* [8] display the energy eigenvalues for general values of the g-factor, and it is clear from their work that, even in the case $g = 1$, i.e. $\gamma = 0$ (no anomalous magnetic moment coupling), the energy eigenvalues take on a far more complicated form. These authors are also able to determine the Shay–Good wavefunctions in a HMF, but it is to be noted that their approach is quite different to that of this paper.

Goldman and Tsai [6] also study the energy eigenvalues obtained via the Shay–Good formalism, and Tsai [21] adds a multispinor formalism result to this work. Goldman *et al* [7] find the energy eigenvalues directly from the generalized Proca vector theory. For the case of $g = 2$ ($\gamma = 1$), these reduce to the result obtained by Krase *et al*, equation (2.5). The only author to study the eigenvalues obtained from the Sakata–Taketani formalism, Weaver [9], finds that in the case of $g = 2$ ($\gamma = 1$), they yield a cubic equation in E^2. As we shall show in the next section, this is due to the use of an incorrect non-minimally coupled Hamiltonian by this author, and elsewhere [22]. Our use of the correct Hamiltonian leads to a canonical result for the energy eigenvalues, and allows a direct evaluation of the eigensolutions.

3. Quantum mechanics of a spin-1 particle in a HMF

3.1. Equations of motion

The Sakata–Taketani Hamiltonian for the case of a relativistic spin-1 particle with a g-factor of 2, in a HMF (of magnitude B and purely in the z direction), can be obtained by setting $\gamma = 1$ and the anomalous quadrupole coupling $q = 0$ in the more general Hamiltonian given by Young and Bludman [16]:

$$\mathcal{H} = e\Phi + \rho_3\left(mc^2 - \frac{e\hbar}{mc}(\boldsymbol{\sigma}\cdot\boldsymbol{B})\right) + \rho_0\frac{1}{2m}\boldsymbol{\pi}^2 - \mathrm{i}\rho_2\frac{1}{m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 \tag{3.1}$$

with Φ the scalar potential, the ρ_i being the Pauli matrices in a representation where ρ_3 is diagonal ($\rho_0 = \rho_3 + \mathrm{i}\rho_2$), and the σ_i are the spin-1 matrices with σ_z diagonal, having the group properties (see section 3 of [1])

$$\sigma_i\sigma_j\sigma_k + \sigma_k\sigma_j\sigma_i = \delta_{ij}\sigma_k + \delta_{jk}\sigma_i \qquad [\sigma_i, \sigma_j] = \mathrm{i}\varepsilon_{ijk}\sigma_k\,. \tag{3.2}$$

The canonical momenta π_i have the commutation relation

$$[\pi_i, \pi_j] = \frac{\mathrm{i}e\hbar}{c}\varepsilon_{ijk}B_k\,. \tag{3.3}$$

The Hamiltonian (3.1) differs from that employed by Weaver [9], which is

$$\mathcal{H} = e\Phi + \rho_3 mc^2 + \rho_0\left(\frac{1}{2m}\boldsymbol{\pi}^2 - \frac{e\hbar}{mc}(\boldsymbol{\sigma}\cdot\boldsymbol{B})\right) - \mathrm{i}\rho_2\frac{1}{m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\,. \tag{3.4}$$

The appropriate normalization for the six-component wavefunction Ψ is given by

$$\int \mathrm{d}^3r\,\Psi^{\epsilon\,\dagger}\rho_3\Psi^{\epsilon} = \epsilon \tag{3.5}$$

where ϵ has a value of $+1$ for positive energy solutions, and -1 for negative energy solutions. As is the usual case for bosons (3.5) indicates a normalization to charge.

Following Johnson and Lippmann [11], we find the time evolution of the momentum and position operators, employing the Heisenberg equations of motion:

$$\dot{\pi}_x = \omega_c\rho_0\pi_y - \omega_c\mathrm{i}\rho_2\sigma_y(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) - \omega_c\mathrm{i}\rho_2(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\sigma_y \tag{3.6}$$

$$\dot{\pi}_y = -\omega_c\rho_0\pi_x + \omega_c\mathrm{i}\rho_2\sigma_x(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) + \omega_c\mathrm{i}\rho_2(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\sigma_x \tag{3.7}$$

$$\dot{\pi}_z = 0 \tag{3.8}$$

with $\omega_c = eB/mc$, and for the operator $\boldsymbol{r}$:

$$m\boldsymbol{r} = \rho_0\boldsymbol{\pi} - \mathrm{i}\rho_2\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) - \mathrm{i}\rho_2(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\boldsymbol{\sigma}\,. \tag{3.9}$$

Combining the equations of motion for the position and momentum operators gives

$$\dot{\pi}_x - m\omega_c\dot{y} = 0 \tag{3.10}$$

$$\dot{\pi}_y + m\omega_c\dot{x} = 0\,. \tag{3.11}$$

Thus, the equations of motion have reduced to the same form as given by Johnson and Lippmann [11] and Witte *et al* [13] for the spin-$\frac{1}{2}$ and spin-0 cases, and thus the equations of motion are identical for particles of up to and including spin 1 (but only when the spin-1 particles have a g-factor of 2, i.e. $\gamma = 1$). The equations of motion imply

$$y_0 = y - \frac{\pi_x}{m\omega_c} \tag{3.12}$$

$$x_0 = x + \frac{\pi_y}{m\omega_c} \tag{3.13}$$

with (x_0, y_0) being the co-ordinates of the centre of gyration. They satisfy the commutation relation

$$[x_0, y_0] = -\frac{\mathrm{i}\hbar c}{eB} = -\mathrm{i}\lambda^2 \tag{3.14}$$

which implies the uncertainty relationship

$$\Delta x_0 \Delta y_0 \geqslant \tfrac{1}{2}\lambda^2\,. \tag{3.15}$$

Here, $\lambda = \sqrt{\hbar c/eB}$ is the Larmor length, a fundamental quantum scale length for a particle in the HMF. The values $r_0^2 = x_0^2 + y_0^2$ and $r_1^2 = (x - x_0)^2 + (y - y_0)^2$ are constants of the motion. This implies that

$$[\pi_\perp^2, \mathcal{H}] = 0 \tag{3.16}$$

where $\pi_\perp^2 = \pi_x^2 + \pi_y^2$, which is identical to the classical result (see [11]).

3.2. Energy eigenvalues

We now obtain the energy eigenvalues, by simply squaring the Hamiltonian (3.1). For that purpose, the following identities are useful:

$$\left[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \left(\pi^2 - \frac{2e\hbar}{c}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right)\right] = 0 \tag{3.17}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^4 = \frac{1}{2}\left\{\pi^2(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 + (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\pi^2 - \frac{2e\hbar}{c}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2(\boldsymbol{\sigma}\cdot\boldsymbol{B}) - \frac{2e\hbar}{c}(\boldsymbol{\sigma}\cdot\boldsymbol{B})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 + \frac{2e\hbar}{c}(\boldsymbol{B}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\right\} \tag{3.18}$$

$$\left\{(\boldsymbol{\sigma}\cdot\boldsymbol{B}), (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\right\} = \left(\pi^2 - \frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right)\boldsymbol{\sigma}\cdot\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\,. \tag{3.19}$$

We obtain

$$\mathcal{H}^2 = \pi^2 c^2 + m^2 c^4 - 2e\hbar c\sigma_z B\,. \tag{3.20}$$

We separate π^2 into components parallel and perpendicular to the magnetic field, and take σ_z to be diagonal, with eigenvalues $s = -1, 0, 1$.

Thus, we have

$$\mathcal{H}^2 = 2mc^2\mathcal{H}_\perp + p_z^2 c^2 + m^2 c^4 - 2e\hbar cBs \tag{3.21}$$

where

$$\mathcal{H}_\perp = \frac{1}{2m}\left(\pi_x^2 + \pi_y^2\right)\,. \tag{3.22}$$

The eigenvalues are those of a quantum-mechanical harmonic oscillator (see [11]). These are

$$\frac{e\hbar B}{mc}\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, 3, \ldots\,. \tag{3.23}$$

As $[\mathcal{H}_\perp, \mathcal{H}^2] = 0$, $\mathcal{H}^2$ and $\mathcal{H}_\perp$ will have simultaneous eigenvectors. Therefore, we obtain for the energy eigenvalues the same result as Krase *et al* [8] and Tsai and co-workers [5–7], equation (2.5). As pointed out earlier, Weaver [9] does not obtain this result, suggesting that the Sakata–Taketani formalism yields different eigenvalues to other formalisms. It is evident that in the particular case of $g = 2$ ($\gamma = 1$), this is not the case.

As can be seen from (2.5), the energy eigenvalues become complex if

$$B > \frac{m^2 c^3}{e\hbar}\,. \tag{3.24}$$

In order to show that this restriction is not too severe, we give an estimate of this critical value, in the case of the particle being a ρ-meson of mass approximately 770 MeV

$$B_{\text{critical}} \simeq 1 \times 10^{20}\ \text{gauss}\,. \tag{3.25}$$

Given that the magnetic field of a pulsar is of the order of 10^{12} gauss, this critical field strength is extremely high, and thus for any physically feasible system, we would suggest that the energy eigenvalues remain real. None the less, theoretical attempts to deal with this pathology have been addressed in previous studies [10, 23].

3.3. Eigenfunctions

The eigenfunctions are chosen to be simultaneous solutions of $\mathcal{H}$, p_z and y_0, thus the centre of gyration is located in the plane $y = y_0$, and x_0 is unspecified. With the asymmetric gauge $A = -\hat{\imath}By$, this choice is equivalent to $p_x = -eBy_0$ being selected as one of the complete set of commuting operators, in order to completely specify the wavefunctions.

The procedure will be similar to that of Johnson and Lippmann [11], where eigenfunctions of $\mathcal{H}$ are constructed from solutions of the eigenfunction equation for $\mathcal{H}^2$, which has simultaneous solutions with $\mathcal{H}_\perp$. We begin with the positive energy solutions.

With the above asymmetric gauge choice, the ladder operators for $\mathcal{H}_\perp$ are

$$\pi_\pm = \pi_x \mp \mathrm{i}\pi_y \,. \tag{3.26}$$

Consider a wavefunction Ψ *that satisfies the eigenvalue equation*

$$\mathcal{H}\Psi = E\Psi \,. \tag{3.27}$$

The wavefunctions may be found by considering solutions of the equation

$$\left(\mathcal{H}^2 - E^2\right)\chi = (\mathcal{H} - E)(\mathcal{H} + E)\chi = 0 \tag{3.28}$$

with

$$\Psi = (\mathcal{H} + E)\chi \,. \tag{3.29}$$

We choose χ to be an eigenfunction of the spin operator σ_z, so that

$$\sigma_z\chi_s = s\chi_s \qquad s = 0, \pm 1 \,. \tag{3.30}$$

We then have the second-order equation

$$\left\{\mathcal{H}_\perp + \frac{1}{2mc^2}\left(p_z^2c^2 + m^2c^4 - 2e\hbar cBs - E^2\right)\right\}\chi_{s,n} = 0 \,. \tag{3.31}$$

To fully specify the solutions $\chi_{s,n}$, we choose to make them eigenfunctions of ρ_3, with eigenvalue 1. Hence

$$\rho_3\chi_{s,n} = \chi_{s,n} \,. \tag{3.32}$$

So, we have

$$\chi_{s,n} = \begin{pmatrix} \varphi_{s,n}\theta_s \\ 0 \end{pmatrix} \tag{3.33}$$

with

$$\theta_{+1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \theta_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \theta_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,. \tag{3.34}$$

In the chosen gauge, the eigenfunctions of $\mathcal{H}_\perp$ are of the form (see reference [13])

$$\varphi_{s,n} = K_{s,n}H_n\left(\frac{1}{\lambda}(y - y_0)\right)\exp\left\{\frac{\mathrm{i}}{\hbar}(xp_x + zp_z) - \frac{1}{2\lambda^2}(y - y_0)^2\right\} \tag{3.35}$$

where H_n is a Hermite polynomial of order n, and the coefficient $K_{s,n}$ is to be determined by normalization.

Employing (3.29), we obtain explicitly the wavefunctions $\Psi_{s,n}$:

$$\Psi_{s,n} = \begin{pmatrix} \left(m^2c^4 + \frac{\pi^2}{2m} - \frac{e\hbar Bs}{mc} + E_{s,n,p_z}\right)\varphi_{s,n}\theta_s \\ \left(-\frac{\pi^2}{2m} + \frac{1}{m}(\sigma\cdot\pi)^2\right)\varphi_{s,n}\theta_s \end{pmatrix}. \tag{3.36}$$

The normalization is given by (3.5), and the negative energy solutions are found by premultiplication of E_{n,s,p_z}, p_x and p_z by a factor of -1, equivalent to changing the sign of the charge in the Proca equations.

We obtain for the eigensolutions $\Psi^{\epsilon}_{s,n}$:

$$\Psi^{\epsilon}_{1,n} = \frac{1}{(\epsilon E_{1,n,p_z} + mc^2)} \frac{\exp\left\{\epsilon\frac{i}{\hbar}(xp_x + zp_z) - \frac{1}{2}\alpha_\epsilon^2\right\}}{\left(\mathcal{L}_x\mathcal{L}_z E_{1,n,p_z} 2^{n+2} n! mc^2 \pi^{\frac{1}{2}}\lambda\right)^{\frac{1}{2}}}$$

$$\times \begin{pmatrix} (\epsilon E_{1,n,p_z} + mc^2)^2 H_n(\alpha_\epsilon) \\ 0 \\ 0 \\ \left(\frac{-\hbar^2c^2}{\lambda^2} + p_z^2c^2\right) H_n(\alpha_\epsilon) \\ \frac{\sqrt{2}}{\lambda}\hbar c^2 \epsilon p_z n H_{n-1}(\alpha_\epsilon) \\ \frac{4}{\lambda^2}\hbar^2c^2 n(n-1) H_{n-2}(\alpha_\epsilon) \end{pmatrix} \tag{3.37}$$

$$\Psi^{\epsilon}_{0,n} = \frac{1}{(\epsilon E_{0,n,p_z} + mc^2)} \frac{\exp\left\{\epsilon\frac{i}{\hbar}(xp_x + zp_z) - \frac{1}{2}\alpha_\epsilon^2\right\}}{\left(\mathcal{L}_x\mathcal{L}_z E_{0,n,p_z} 2^{n+2} n! mc^2 \pi^{\frac{1}{2}}\lambda\right)^{\frac{1}{2}}}$$

$$\times \begin{pmatrix} 0 \\ (\epsilon E_{0,n,p_z} + mc^2)^2 H_n(\alpha_\epsilon) \\ 0 \\ \frac{\sqrt{2}}{\lambda}\hbar c^2 \epsilon p_z H_{n+1}(\alpha_\epsilon) \\ \left(\frac{\hbar^2c^2}{\lambda^2}(2n+1) - p_z^2c^2\right) H_n(\alpha_\epsilon) \\ \frac{-2\sqrt{2}}{\lambda}\hbar c^2 \epsilon p_z n H_{n-1}(\alpha_\epsilon) \end{pmatrix} \tag{3.38}$$

$$\Psi^{\epsilon}_{-1,n} = \frac{1}{(\epsilon E_{-1,n,p_z} + mc^2)} \frac{\exp\left\{\epsilon\frac{i}{\hbar}(xp_x + zp_z) - \frac{1}{2}\alpha_\epsilon^2\right\}}{\left(\mathcal{L}_x\mathcal{L}_z E_{-1,n,p_z} 2^{n+2} n! mc^2 \pi^{\frac{1}{2}}\lambda\right)^{\frac{1}{2}}} \times$$

$$\times \begin{pmatrix} 0 \\ 0 \\ (\epsilon E_{-1,n,p_z} + mc^2)^2 H_n(\alpha_\epsilon) \\ \frac{\hbar^2 c^2}{\lambda^2} H_{n+2}(\alpha_\epsilon) \\ \frac{-\sqrt{2}}{\lambda}\hbar c^2 \epsilon p_z H_{n+1}(\alpha_\epsilon) \\ \left(\frac{\hbar^2 c^2}{\lambda^2} + p_z^2 c^2\right) H_n(\alpha_\epsilon) \end{pmatrix} \tag{3.39}$$

with $\alpha_\epsilon = (1/\lambda)(y - \epsilon y_0)$, and where $\mathcal{L}_x$ and $\mathcal{L}_z$ are lengths of the system in the x and z directions respectively.

3.4. Current

The six-component current is obtained by taking the vector current of Corben and Schwinger [4], and using a procedure developed by Young and Bludman [16], by which they obtain the generalized six-component Hamiltonian (with the inclusion of anomalous moments) from the generalized Proca equations. This is done by explicit elimination of the dynamically redundant components of the spin-1 fields, as Pierce decompostion of the Duffin–Kemmer formalism for the case where anomalous moments are included is considerably more intricate.

The vector current given in [4] is

$$j_\mu = \mathrm{i}e\left[U_\nu^\dagger U_{\mu\nu} - U_{\mu\nu}^\dagger U_\nu + \gamma\partial_\nu\left(U_\mu^\dagger U_\nu - U_\nu^\dagger U_\mu\right)\right] \tag{3.40}$$

where U_μ is the vector field, and $U_{\mu\nu} = \pi_\mu U_\nu - \pi_\nu U_\mu$.

The terms multiplying γ are those due to the anomalous magnetic moment. Setting $\gamma = 1$ gives $g = 2$. The six-component current we obtain from transforming (3.40) is

$$\begin{aligned} \jmath_i = \frac{e}{2m}\Psi^\dagger \Big\{ & \rho_3(1+\rho_1+2\rho_0)\pi_i - \rho_3(1+\rho_1+\rho_3+2\mathrm{i}\rho_2)[\sigma_i(\sigma\cdot\pi) + (\sigma\cdot\pi)\sigma_i] \\ & + \frac{\mathrm{i}}{2}(5\rho_3 - \mathrm{i}\rho_2 + \rho_1 + 1)(\pi\times\sigma)_i + \frac{\mathrm{i}e}{mc}\rho_3(1+\rho_1)\mathrm{E}_i \\ & + \frac{1}{m^2c^2}\rho_3\left[(1+\rho_1)\frac{\mathrm{i}e\mathrm{E}_i}{mc} - \rho_0\pi_i^\dagger\right]\left[\pi^2 - (\sigma\cdot\pi)^2\right] \\ & - \frac{e\hbar^2}{m^2c^4}\rho_3(\rho_3 - \mathrm{i}\rho_2)\left[(\nabla\times \boldsymbol{B})_i + \frac{\mathrm{i}}{2}(\sigma\times(\nabla\times\boldsymbol{B}))_i\right] \\ & + \frac{e\hbar^2}{2m^2c^4}\rho_3(\rho_3 - \mathrm{i}\rho_2)[\sigma_i\sigma\cdot(\nabla\times\boldsymbol{B}) + \sigma\cdot(\nabla\times\boldsymbol{B})\sigma_i] \\ & + \frac{e\hbar}{m^3c^4}\Big\{\rho_3\left[(\rho_3 - \mathrm{i}\rho_2)\frac{\epsilon\mathrm{E}_i}{mc} - \mathrm{i}(1-\rho_1)\pi_i\right] \\ & \times[(\sigma\cdot\pi)(\sigma\cdot\boldsymbol{E}) - \mathrm{i}((\pi\times\boldsymbol{E})\cdot\sigma - (\pi\cdot\boldsymbol{E}))]\Big\} \end{aligned}$$

$$- \frac{e\hbar^2}{m^3c^4}\rho_3 (\nabla \cdot \boldsymbol{E}) \left[(\rho_3 - \mathrm{i}\rho_2) \frac{\mathrm{i}eE_i}{mc} - (1+\rho_1)\pi_i \right]$$
$$- \frac{e\hbar^2}{2m^3c^4}\rho_3 (\nabla \cdot \boldsymbol{E}) \left\{ (1+\rho_1) [\sigma_i (\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) + (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\sigma_i] \right.$$
$$\left. - \frac{\mathrm{i}e}{mc}(\rho_3 - \mathrm{i}\rho_2)[\sigma_i(\boldsymbol{\sigma}\cdot\boldsymbol{E}) + (\boldsymbol{\sigma}\cdot\boldsymbol{E})\sigma_i] \right\}$$
$$+ \frac{\mathrm{i}e\hbar^2}{2m^3c^4}\rho_3 (\nabla \cdot \boldsymbol{E}) \left[(1+\rho_1)(\boldsymbol{\pi}\times\boldsymbol{\sigma})_i - \frac{\mathrm{i}e}{mc}(\rho_3 - \mathrm{i}\rho_2)(\boldsymbol{E}\times\boldsymbol{\sigma})_i \right]$$
$$- \frac{e^2\hbar^2}{m^2c^4}\rho_3(\rho_3 - \mathrm{i}\rho_2) \left[\partial_t \mathrm{E}_i + \frac{\mathrm{i}}{2}((\partial_t \boldsymbol{E})\times\boldsymbol{\sigma})_i \right]$$
$$\left. + \frac{e^2\hbar^2}{2m^2c^4}(\rho_3 - \mathrm{i}\rho_2)[\sigma_i(\boldsymbol{\sigma}\cdot(\partial_t\boldsymbol{E})) + (\boldsymbol{\sigma}\cdot(\partial_t\boldsymbol{E}))\sigma_i] \right\} \Psi + \mathrm{HC} \qquad (3.41)$$

and where $\boldsymbol{E}$ is the electric field.

4. Application to a pair plasma in a HMF

Previous studies have employed the self-consistent RPA method for the determination of the modes of oscillation of particle–antiparticle plasmas in the cases spin-0 [14, 15], spin-$\frac{1}{2}$ [15] and, for no external fields, spin-1 [1] particles. The wavefunctions Ψ of section 3, equations (3.37)–(3.39), and the current (3.41), could similarly be employed in a study of a spin-1 pair plasma in a HMF.

The first step would be to second-quantize the wavefunctions and relevent operators. The second-quantized field is

$$\hat{\Psi} = \sum_{s,n,p_z} \left\{ b_{s,n,p_z}(t)\Psi^+_{s,n,p_z}(\boldsymbol{r}) + d^\dagger_{s,n,p_z}(t)\Psi^-_{s,n,p_z}(\boldsymbol{r}) \right\} \qquad (4.1)$$

where b_{s,n,p_z} and d_{s,n,p_z} are respectively the destruction operators for a particle and an antiparticle state.

Operators, specifically the Hamiltonian (3.1) and current (3.41), are second-quantized by the procedure

$$\hat{\mathcal{O}} = \int \mathrm{d}^3 r\, \hat{\Psi}^\dagger \rho_3 \mathcal{O} \hat{\Psi}\,. \qquad (4.2)$$

The current (3.41) would include both electric and magnetic fields in an expansion to a first-order perturbative potential $A^\mu_1(\boldsymbol{r}, t)$, due to interactions within the plasma itself. Similarly, the full Hamiltonian, including coupling to electric and magnetic fields, is required, and is given by Young and Bludman [16]. For a g-factor of 2 ($\gamma = 1$), it is

$$\mathcal{H} = e\Phi + \rho_3\left(mc^2 - \frac{e\hbar}{mc}(\boldsymbol{\sigma}\cdot\boldsymbol{B}) \right) + \rho_0 \frac{1}{2m}\pi^2 - \mathrm{i}\rho_2 \frac{1}{m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2$$
$$- \frac{\mathrm{i}e\hbar}{2m^2c^2}(1+\rho_1)[(\boldsymbol{\sigma}\cdot\boldsymbol{E})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) - \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{\pi}) - \boldsymbol{E}\cdot\boldsymbol{\pi}]$$
$$+ \frac{\mathrm{i}e\hbar}{2m^2c^2}(1-\rho_1)[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{E}) - \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E}) - \boldsymbol{\pi}\cdot\boldsymbol{E}]$$
$$- \frac{e^2\hbar^2}{2m^3c^4}(\rho_3 - \mathrm{i}\rho_2)\left[(\boldsymbol{\sigma}\cdot\boldsymbol{E})^2 - E^2 \right]. \qquad (4.3)$$

The linearization in the perturbative potential $A_1^\mu(\boldsymbol{r}, t)$ gives the linearized interaction Hamiltonian $\mathcal{H}_1$, and second-quantization its operator form. This would allow the determination of the equations of motion of the linearized boson and antiboson wavefunctions. Second-quantization of the current would then allow extraction of the polarization tensor via

$$J^\nu(k) = \sum_{k'} \Pi^\nu{}_\mu(k, k') A^\mu(k') \,. \tag{4.4}$$

Note that the relationship is one in Fourier space, and thus the potential and current must be Fourier-transformed.

Once the polarization tensor is extracted, then a mode analysis of the plasma similar to that in [1] or [14] can be done. Considering the complexity of the Hamiltonian (4.3), and more so the overwhelming intricacy of the current (3.41), we do not propose to carry out the full procedure involved, but present this brief outline to show that the problem could certainly be attempted in theory, and is a direct application of the spin-1 boson quantum mechanics we have developed in this paper. It is, however, evident that a full study of a spin-1 pair plasma in a HMF, as mooted in [1], is a goal to be achieved only through an inordinately arduous calculation.

5. Conclusion

In section 2 of this paper, we gave a thorough outline of previous work in the development of the quantum mechanics of spin-1 bosons in a HMF, and discussed the eigenvalues obtained for this system via differing spin-1 formalisms, with particular reference to the case where the g-factor is 2 ($\gamma = 1$). In section 3, we developed the quantum mechanics of this system, employing the six-component formalism of Sakata and Taketani, giving the energy eigenvalues and the wavefunctions for both the boson and antiboson cases. We also derived the six-component current for the general case of external electric and magnetic fields. In section 4, we discussed how our results of section 3 could be employed in a study of the relativistic boson–antiboson plasma in an external HMF, the generalization of work presented previously in [1] on the spin-1 boson plasma in no external fields.

Acknowledgments

The authors are indebted to V Kowalenko and C P Dettmann for fruitful discussions and assistance during the course of this work. One of us (JD) would like to acknowledge the support of the Australian Postgraduate Research Award Program.

References

[1] Daicic J and Frankel N E 1992 *Prog. Theor. Phys.* **88** 1
[2] Proca A 1936 *J. Physique Rad.* **7** 347
[3] Sakata S and Taketani M 1940 *Proc. Phys.-Math. Soc. Japan* **22** 757
[4] Corben H C and Schwinger J 1940 *Phys. Rev.* **58** 953

[5] Tsai W and Yildiz A 1971 *Phys. Rev.* D **4** 3643
[6] Goldman T and Tsai W 1971 *Phys. Rev.* D **4** 3648
[7] Goldman T, Tsai W and Yildiz A 1972 *Phys. Rev.* D **5** 1926
[8] Krase L D, Lu P and Jr Good R H 1971 *Phys. Rev.* D **3** 1275
[9] Weaver D L 1976 *Phys. Rev.* D **14** 2824; 1978 *J. Math. Phys.* **19** 88; 1978 *Am. J. Phys.* **46** 721
[10] Vijayalakshmi B, Seetheraman M and Mathews P M 1979 *J. Phys. A: Math. Gen.* **5** 665
[11] Johnson M H and Lippmann B A 1949 *Phys. Rev.* **76** 828
[12] Durand E 1975 *Phys. Rev.* D **11** 3405
[13] Witte N S, Dawe R L and Hines K C 1987 *J. Math. Phys.* **28** 1864
[14] Witte N S, Kowalenko V and Hines K C 1988 *Phys. Rev.* D **38** 3667
[15] Kowalenko V, Frankel N E and Hines K C 1985 *Phys. Rep.* **126** 109
[16] Young J A and Bludman S A 1963 *Phys. Rev.* **131** 2326
[17] Heitler W 1943 *Proc. R. Irish Acad.* **49** 1
[18] Duffin R J 1938 *Phys. Rev.* **54** 1114
Kemmer N 1939 *Proc. R. Soc.* A **173** 91
[19] Velo G and Zwangzinger D 1969 *Phys. Rev.* **188** 2218
[20] Shay D and Good R H 1969 *Phys. Rev.* **179** 1410
[21] Tsai W 1971 *Phys. Rev.* D **4** 3652
[22] Roux J F 1984 *Lett. Nuovo Cimento* **40** 63
[23] Nielsen N K and Olsen P 1978 *Nucl. Phys.* B **144** 376